Chaos in nonpropagating hydrodynamic solitons

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The time series of the amplitude of hydrodynamic solitons have been measured and investigated in a Faraday experiment. The Lyapunov exponent spectra prove that the soliton waves always evolve with chaotic behavior. The distributions of the exponents over the plane of the driving parameters show that large exponents correspond to small stability of solitary waves. The experimental results also predict that there may exist some chaotic equations in the high order reductions of the fundamental hydrodynamical equations together with the integrable nonlinear Schrödinger equation in low order. [S1063-651X(96)09405-5]

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I. INTRODUCTION

During the last decade the investigation of the nonlinear dynamics in Faraday experiments [1] has proved to be extremely fruitful for understanding nonlinear systems in general. Many experimental systems and theoretical models in water [2], in lattices [3], and even in granular materials [4] have been of fundamental importance for achievements in these fields. The theoretical work [5,6,3] on this topic is done in the frame of integrable systems, such as the cubic nonlinear Schrödinger (NLS) equations [7]. At the same time, much progress has been made concerning another kind of nonlinear phenomena, say, chaos. Chaos is a state in which a nonlinear dynamical system exhibits bounded motion with exponential sensitivity to initial conditions, in that initially neighboring states of a chaotic system diverge exponentially (on average) as the system evolves forward in time [8]. A number of chaotic time series from both the experiments and numerical procedures have been investigated in various fields. A great deal of attention, however, has been focused only on either the integrable approach or the unintegrable approach, with two quite different methods. Fortunately, one has also found the transition from integrable systems to chaos in experimental observations [9] and mathematical theories [10,11]. Recently, a Faraday experiment on water in a small rectangular tank shows that nonpropagating hydrodynamic solitons manifest a chaotic behavior when subjected to a periodically modulated excitation [12]. In this paper, we report the careful measurement of 19 time series of hydrodynamic soliton amplitudes, and prove that a spatially solitary modulation evolves with chaotic behavior in a Faraday experiment on water subjected to a simple harmonic (sinusoidal) excitation by using the method of the Lyapunov exponent spectrum.

II. EXPERIMENTAL SETUP AND MEASUREMENT

The experimental apparatus depicted in Fig. 1 consists of two parts, the vibrating part and the measuring one. In the former the exciter (Brüel & Kjaer 4812) is driven by a sinusoidal voltage generated and amplified by the exciter control (Brüel & Kjaer 1050) and the power amplifier (Brüel & Kjaer 2707), respectively. A rectangular waveguide is attached on the aluminum table of the exciter. The motion of the table is measured and controlled by the exciter control according to the signal detected by an accelerometer (Brüel & Kjaer 4393) fixed at the table. All motion signals are pure enough to neglect the effect of the harmonics on the results; for example, the second harmonic is 40 dB lower than the fundamental. The data of the wave forms are generated by a transducer consisting of a pair of parallel thin metal probes with 1 mm separation. To prevent errors caused by the electrolysis of water, we connect two probes to a source of 20 kHz and 10 V through a resistor. The modulating voltage across the resistor is fed to a lock-in analyzer (EG&G 2504) for demodulation and amplification, then to a data acquisition and control unit (HP 3852A) for acquisition, and finally to a controlling computer for storage, and then proceeds through the Hewlett-Packard Interface Bus. As the probes are put into water with weak electrical conductivity, the electrical resistance or modulated voltage for a current flow through the resistor will be approximately proportional to the height of the surface wave. The experimental results are also recorded by both a video and a photographic camera. The waveguide used is 29.8 mm in width and 198 mm in length and is filled by water up to a depth of 18 mm.

Under a suitable excitation of frequency $2f_e$ and amplitude A_e, a nonpropagating hydrodynamical soliton (breather)

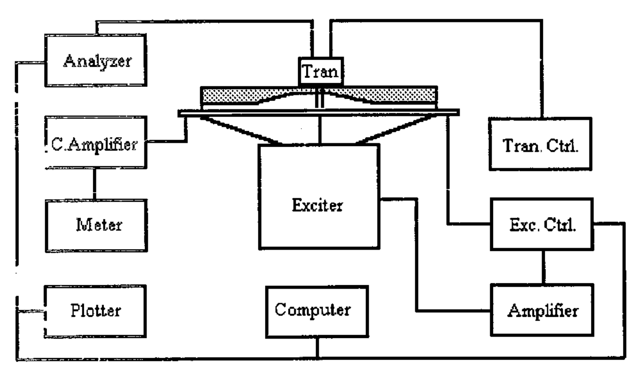

FIG. 1. The scheme of the experimental apparatus.

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in the (0,1) mode can be formed and then becomes robust [2]. We fix the transducer just at the position of the soliton peak (antinode) and set the data acquisition in the state of the voltage scan for a single channel with the interval 0.002 sec, about 100 data points each period of the (0,1) mode. For every one of 19 sets of $2f_e$ and A_e, the acquisition procedure lasts about 20 sec, say, 10 000 points. Finally, we obtain in total 19 experimental time series of 190 000 data points.

III. LYAPUNOV EXPONENTS

Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in phase space and they are fundamental indicators of chaos. Positive exponents mean the divergence of nearby orbits in phase space; moreover, the more positive the exponents, the higher the orbit divergence. The theoretical method for the calculation of the Lyapunov exponents, however, cannot usually be applied to experimental data directly. Fortunately, Wolf *et al.* [13] and Holzfuss and Lauterborn [14] have developed their algorithms for the Lyapunov exponents from an experimental time series. The method described by Holzfuss and Lauterborn not only can be applied directly to calculate the exponents but also avoids the effect of spurious exponents, which are more negative than the most negative true exponent because the true exponents indeed converge to their asymptotic values with the increase of the embedding dimension.

In the method of calculation of Ref. [14], the Lyapunov exponents are expressed by

$$\lambda_i = \lim_{k\to\infty} \frac{1}{k\Delta t} \sum_{j=0}^{k-1} \ln r_{ii}^j , \tag{1}$$

where r_{ii}^j are the diagonal elements of the $n\times n$ matrix R_j which is an upper triangular matrix with non-negative diagonal elements, with n being the embedding dimension (matrix dimension), and Δt is the evolution time with $k\Delta t = t$ being the total time of the experimental series. R_j can be obtained by QR [15] decomposing the $n\times n$ matrix A_j defined as

$$A_j \equiv CV^{-1} \quad \text{with} \quad (C)_{kl} = \frac{1}{N}\sum_{i=1}^{N} Z_k^i Y_l^i ,$$

$$(V)_{kl} = \frac{1}{N}\sum_{i=1}^{N} Y_k^i Y_l^i , \tag{2}$$

where Y^i $(i=1,\ldots,N)$ are N difference vectors in a sphere with a small radius ϵ centered at the phase point at time t. Z^i $(i=1,\ldots,N)$ are the map vectors of Y^i $(i=1,\ldots,N)$ after an evolution time Δt. Before calculating the exponents we reconstruct the signal time series into n-dimensional time series by using the time delay $t_d = 10t_s$ with t_s being the sampling time, 0.002 sec here. In the calculation we choose the evolution time $\Delta t = 10t_s$, and the radius ϵ of the n-dimensional sphere needed for the approximation of the matrix A_j is kept as small as possible, and is enlarged whenever the number of data points N inside a sphere is less than $n+2$ or a singularity arises in the procedure of QR decomposition.

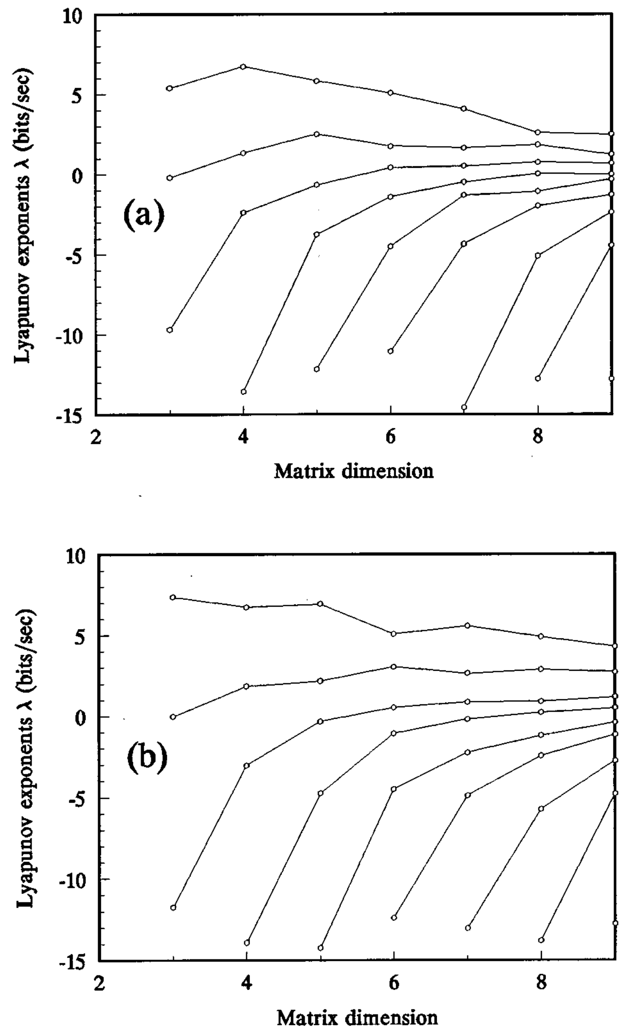


FIG. 2. The Lyapunov exponent spectra vs the matrix dimension. The driving parameters are (a) $2f_e = 9.3$ Hz and $A_e = 0.75$ mm, (b) $2f_e = 9.1$ Hz and $A_e = 1.00$ mm.

Figure 2 shows the Lyapunov exponents for matrix dimensions from 3 to 9. The driving frequency $2f_e$ and the amplitude A_e are (a) 9.3 Hz and 0.75 mm, and (b) 9.1 Hz and 1.00 mm, respectively. With increasing matrix dimension the largest exponents converge to their asymptotic values of (a) $\lambda_1 = 2.5$ bits/sec and (b) $\lambda_1 = 4.2$ bits/sec, and the second largest ones reach (a) $\lambda_2 = 1.2$ bits/sec and (b) $\lambda_2 = 2.7$ bits/sec, respectively.

The Lyapunov exponent spectra show that the amplitudes of the waves in the Faraday experiment evolve with chaotic behavior though the spatial modulation is solitary in the longitudinal direction of the waveguide. Furthermore, the calculations for 17 other sets of driving parameters confirm the conclusions above. Figure 3 shows the projections of the data of Fig. 2 in a two-dimensional subspace of the eight-dimensional reconstructed space. Comparing Figs. 3(a) and 3(b), we can see that time series with small positive exponents correspond to a slight dispersion of the strange attractor, and vice versa.

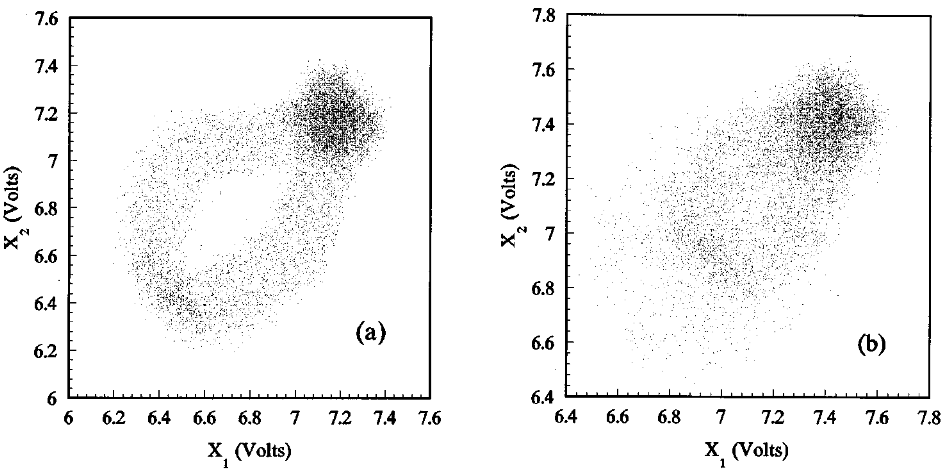


FIG. 3. The two-dimensional projections of the eight-dimensional embedding attractors with time delay 0.02 sec. (a) A weak chaos (the largest Lyapunov exponent λ_1=2.5 bits/sec); (b) a strong chaos (λ_1=4.2 bits/sec). The driving parameters are the same as those in Figs. 2(a) and 2(b), respectively.

IV. STABILITY REGION

For a given waveguide of width b filled with water up to depth d, the nonpropagating hydrodynamic soliton in the Faraday experiment only exists permanently in a certain region in the two-dimensional plane of driving parameters $2f_e$ and A_e. This region in the parametric plane is called the stability region of the soliton [9]. The experiment [2] showed that the soliton appeared only at $f_e < f_0$ with f_0 being the (0,1) modal eigenfrequency of water in the waveguide. We also find that the stability region of a soliton pair with phase mismatch may overlap partially with that of the single soliton. The driving amplitude for a soliton pair, of course, is usually larger than that for a (single) soliton. Figure 4 shows the stability region for a soliton, S_1, and a soliton pair with phase mismatch, S_2. The 19 sets of parameters for which we measure the time series, of course, are located in S_1, that is, A_e=0.75, 0.875, and 1.00 mm at 6–7 frequencies from 8.80 to 10.00 Hz (see the circles in Fig. 4). The first and second largest Lyapunov exponents in the eight-dimensional reconstructed space are plotted as functions of these driving parameters in Figs. 5(a) and 5(b), respectively. Although the number of data sets is still not enough to draw three-dimensional surfaces of the Lyapunov exponents over the parametric plane, we can also conclude the following. (i) The chaotic evolution of the solitary wave always exists everywhere in the stability region. (ii) For both small and large driving amplitudes, say, A_e=0.750 and 1.00 mm, λ_1 and λ_2 decrease at the central frequencies and increase near the low frequency margin through which the (0,1) modal solitons disappear, but similar peaks seem not to occur on the high frequency margin through which the (0,1) modal solitons are replaced by other (0,1) modal solitons, namely, soliton pairs. (iii) For A_e=0.875 mm the relationship between λ's and $2f_e$ is not clear.

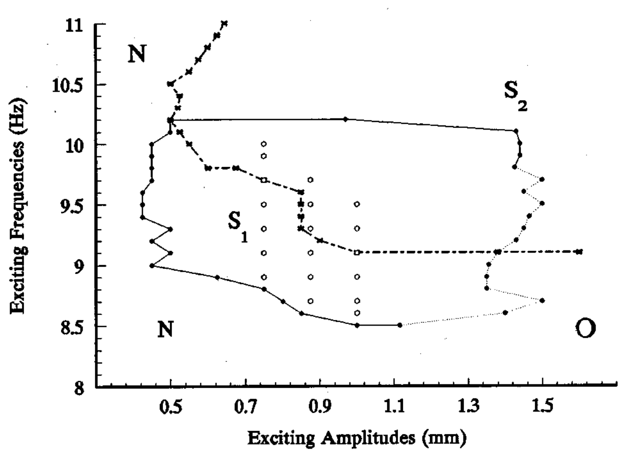


FIG. 4. The stability regions of the (0,1) modal solitary waves for waveguide width 29.5 mm and water depth 18 mm. S_1 is the region for the single soliton, S_2 for the soliton pair, N is the still region, and O for other waves. The dotted curve denotes the margin between S_1 and O.

Based on the phenomena mentioned above, the chaotic behaviors of solitons are related to deviation from the (0,1) modal wave motion instead of solitary motion. In other words, the Lyapunov exponents represent the degree of mode competition rather than the competition between the single soliton and soliton pair. The (0,1) modal soliton, of course, cannot exist if the (0,1) modal waves disappear. Therefore the Lyapunov exponents we calculated also denote the stability of the (0,1) modal soliton in this meaning. The smaller the exponents, the greater the stability of the (0,1) modal soliton.

On the other hand, the wave forms of the hydrodynamic solitons are also related to the Lyapunov exponents. Figures 6(a) and 6(b) show the experimental photographs for the small and large Lyapunov exponents of Figs. 2(a) and 2(b), respectively. It is easy to see that the solitary modulation in the horizontal direction has been distorted so seriously that the sectional curve lacks smoothness for a large exponent [see Fig. 6(b)], which further confirms the concluded relation between the exponents and the (0,1) modal stability.

V. ANALYSIS AND DISCUSSION

Since the nonpropagating hydrodynamic soliton was first discovered in 1984 [2], Larraza and Putterman [5] and Miles [6] have published their theories of the nonlinear surface waves of liquids. The surface displacement $z=\eta(x,y,t)$ with x, y, and z axes being parallel to the length, width, and height directions of the rectangular waveguide, was obtained by perturbation methods. Usually, only the first-order approximation of η could be explicitly expressed as

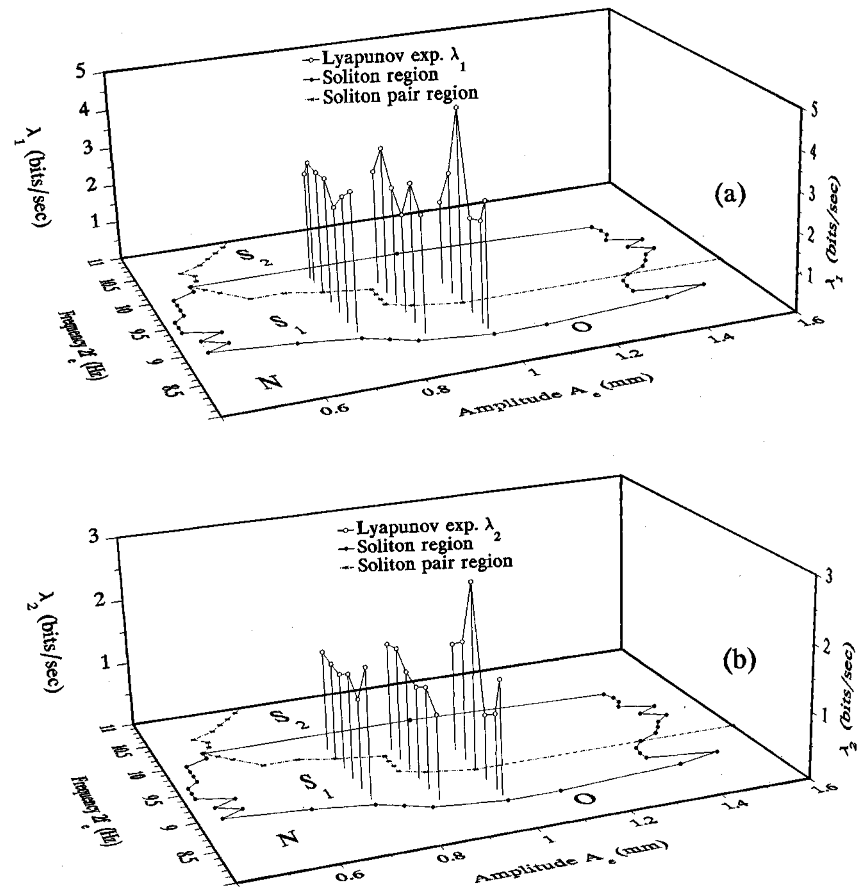


FIG. 5. The distributions of the Lyapunov exponents over the driving parameter plane. (a) λ_1; (b) λ_2. The waveguide width is 29.5 mm and water depth is 18 mm.

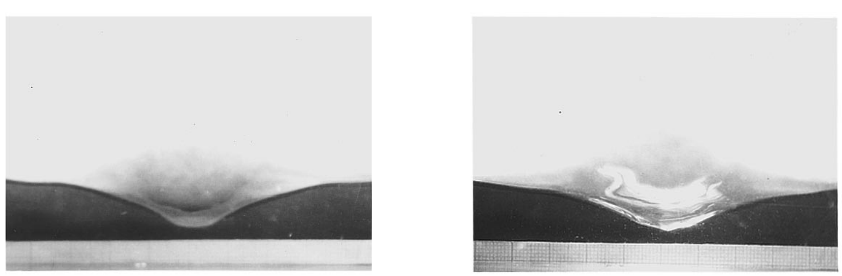
$$\eta^{(1)} = u(x,t)\cos(ky)e^{i\omega t} + \text{c.c.}, \tag{3}$$

where c.c. is the complex conjugate, the (0,1) modal wave number $k=\pi/b$, the (0,1) modal angular frequency $\omega=[gk\,\tanh(kd)]^{1/2}$, and $u(x,t)$ satisfies a NLS equation. One can explain nonpropagating solitary waves, including breather solitons and kink solitons, easily in the framework of theory of the NLS equation. Unfortunately, up to now the higher order solutions for η still remain unknown. Furthermore, the form of the NLS equation can also be different for different order estimations of the parameters. On the other hand, mathematical procedures have shown that some integrable models such as the Korteweg–de Vries (KdV) system can transform themselves into chaotic systems as some terms are added to the models [10]. Therefore the phenomena reported here show that there should exist some chaotic equations among the equations reduced from the fundamental hydrodynamic equations with boundary conditions. In other words, the precise reductions from the fundamental equations are really worthy of further investigation.

At the same time, the experimental data also provide some interesting phenomena not yet explained, such as why the maximum peaks of the λ's are located near the central frequency zone of $A_e=1.00$ mm.

VI. SUMMARY

In a parametrically excited Faraday experiment, the amplitude time series of nonpropagating hydrodynamical solitons have been measured. The Lyapunov exponent spectra of the reconstructed data show that the solitons evolve with chaotic behavior. So the observations reported here indicate that spatial solitary modulation and temporal chaotic evolu-



(a) (b)

FIG. 6. Photographs of soliton waves. (a) Weak chaos; (b) strong chaos. The driving parameters are the same as those in Figs. 2(a) and 2(b), respectively.

tion can exist in the same physical system. In terms of mathematics, both integrable and unintegrable models can be obtained simultaneously by reducing the same fundamental equations with their boundary conditions. Furthermore, the relation between the Lyapunov exponents and the stability of the soliton has been discussed by analyzing the distributions of the exponents over the driving parameter plane. It has been concluded that the smaller the exponents, the stronger the stability of the (0,1) modal waves. The experimental results predict that there may exist some high order chaotic solutions of the surface displacement together with the (0,1) modal soliton in first order. Therefore it is necessary theoretically to reduce precisely the fundamental hydrodynamical equations of water in a waveguide subjected to parametric excitation. Further experiments on the temporal evolution behaviors in other solitary waves together with the corresponding theories are worthy of further investigation. We are now in the process of several experiments for these purposes, which are very encouraging, and the results will be reported later.

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